The BV-BFV Formalism

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Outline

BV and BFV

- Both the (classical) BV and BFV formalism are described in terms of a solution of the classical master equation (CME) $\{S, S\} = 0$, where S and $\{\ ,\ \}$ have opposite parity:
  - **BV** S even and of ghost number 0, $\{\ ,\ \}$ odd and of ghost number 1
  - **BFV** S odd and of ghost number 1, $\{\ ,\ \}$ even and of ghost number 0

  In both cases, the BRST operator $Q := \{S,\ \}$ is odd and of ghost number 1.**P**
- Other degrees may be considered. For example, multivector fields on a manifold M, i.e., sections of $\Lambda^\bullet TM$ (duals of differential forms), have a natural odd bracket of degree -1 extending the Lie bracket of vector fields: the Schouten–Nijenhuis bracket.
- To get Q as above, S must now be even and of degree 2: a bivector field. It will solve the classical master equation iff it is a Poisson bivector field on M.

Some recollections

- The CME in BV is an approximation of the quantum master equation (QME)
$$\Delta e^{\frac{i}{\hbar}S} = 0$$
or equivalently
$$\frac{1}{2}(S, S) - i\hbar\Delta S = 0,$$
where Δ is the BV Laplacian. **P**
The integral of $e^{\frac{i}{\hbar}S}$ on a "Lagrangian submanifold of the BV space" (gauge fixing) is invariant under deformations.**P**
- The BFV action contains information about first-class constraints (a coisotropic submanifold) of some symplectic manifold. **P**In the regular case, the cohomology of Q in ghost number zero yields the Poisson algebra of functions of the symplectic reduction
$$\{\text{zero locus of the constraints}\}/\{\text{their Hamiltonian vector fields}\}$$
This is, e.g., the reduced phase space in the Hamiltonian formulation of a field theory.

Differential graded symplectic description

- We prefer using a symplectic structure instead of the induced Poisson bracket.
- Our general data will be
  - a symplectic structure ω of ghost number $n-1$
  - an "action" S of ghost number n
  - a vector field Q of ghost number 1 satisfying

$$\iota_Q \omega = \mathrm{d}S$$

  with parities equal to ghost number modulo 2.**P**
- The classical master equation implies

$$[Q, Q] = 0$$

  and is equivalent to it for $n \neq -1$. Such a vector field is called a cohomological vector field. **P**
- We call the above collection of data a BF^{n}V structure.

Examples of BF^{n}V structures

There are three important particular cases:

- $n = 0$ This is the Batalin–Vilkovisky (BV) formalism used in QFT. **P**
- $n = 1$ This is the Batalin–Fradkin–Vilkovisky (BFV) formalism used to give a cohomological resolution of symplectic reduction.**P**
- $n = 2$ If there are only coordinates of nonnegative degree, this is a Poisson structure.**P**
  More generally, it describes Poisson structures up to homotopy (i.e., the Poisson bracket is an L_∞-structure).**P**

$$\text{Poisson}_\infty = \text{BF}^2\text{V}$$

Example: gauge theories

- Suppose we have a gauge theory with space of fields F_M, a local action functional S_M^0 and a gauge group acting on it.
- We denote by ϕ the fields, by c the ghosts and by δ_{BRST} the BRST operator. We assign degrees by $|\phi| = 0$ and $|c| = 1$.
- We introduce antifields ϕ^+, c^+ with opposite parity than the field and degrees given by $|\phi^+| = -1$ and $|c^+| = -2$.
- We denote by $\mathcal{F}_M$ the space of all the (ϕ, c, ϕ^+, c^+)'s and we set

$$S_M = S_M^0(\phi) + \int_M (\phi^+ \, \delta_{\text{BRST}}\phi + c^+ \, \delta_{\text{BRST}}\, c),$$

$$\omega_M = \int_M (\delta\phi^+\delta\phi + \delta c^+\delta c)\mathbf{P}$$

- Then, if $\partial M = \emptyset$, the BV action S satisfies the CME. **P**
- One usually considers these “BRST-like” theories, but there are more general examples.

Relaxed structures

- Suppose we have a graded manifold M with a cohomological vector field Q, a function S of degree $n \geq 0$ and a closed 2-form ω of degree $n-1$. **P**We set
$$\check{\alpha} := \iota_Q \omega - \mathrm{d}S$$
and
$$\check{\omega} := \mathrm{d}\check{\alpha} = -\mathrm{L}_Q\,\omega$$
- It turns out that $\check{\omega}$ is a closed, Q-invariant 2-form of degree n. **P**
- We denote by $\underline{M}$ the quotient of M by the kernel of $\check{\omega}$ (assume it is smooth). We denote by $\underline{\omega}$ its symplectic form of degree n.
- It turns out that Q is projectable to a cohomological vector field $\underline{Q}$, which is automatically Hamiltonian ($\iota_{\underline{Q}}\underline{\omega} = \mathrm{d}\underline{S}$). Therefore,
$\underline{M}$ becomes a BF^{n+1}V manifold. **P**
- If $\check{\alpha}$ also descends to $\underline{M}$, we have $\underline{\omega} = \mathrm{d}\underline{\alpha}$ and the modified classical master equation (mCME)
$$\iota_Q \omega = \mathrm{d}S + \pi^* \underline{\alpha}$$

Field theory

- Suppose that M is a space of fields on some closed manifold Σ. **P**
- Suppose we have a BF^{n}V structure on M with ω, Q, and S local.
- This allows us to write ω, Q, and S also on some Σ with boundary.**P**
- If S contains derivatives of the fields, there will be boundary terms that spoil the structure.
- This relaxed structure will however induce a BF^{n+1}V structure on the fields on $\partial\Sigma$ (the kernel of $\breve{\omega}$ contains in particular fields in the bulk).

An application

- Example $n = 0$. On Σ we have a (relaxed) BV structure describing the symmetry content of some field theory. On $\partial\Sigma$ we have a BFV structure describing its reduced phase space (possibly up to homotopy).
- Example $n = 1$. On Σ we have a (relaxed) BFV structure describing the reduced phase space of some field theory. On $\partial\Sigma$ we have a BF^{2}V structure describing a Poisson structure (possibly up to homotopy): the "current algebra" of the theory.**P**
- Under some regularity assumptions, all this is compatible with cutting and gluing of space–times manifolds with corners. **P**
- In several cases, it is also possible to quantize these pictures.

Example: Chern–Simons theory

- Let Σ be a 2-manifold and $\mathfrak{g}$ a quadratic Lie algebra.
- Let N be the space of $\mathfrak{g}$-valued 1-forms A (connections) on Σ with the Atiyah–Bott symplectic structure $\omega = \frac{1}{2}\int_\Sigma \delta A \delta A$.**P**
- We let C denote the space of flat connections. Then $\underline{C}$ turns out to be the quotient by gauge transformations.**P**
- BFV: $M = N \times T^*(\Omega^0(\Sigma, \mathfrak{g})[1])$ and

$$S = \int_\Sigma (c, F_A) + \frac{1}{2}(b, [c, c])\mathbf{P}$$

- On $\partial\Sigma$ we get $\underline{\omega} = \int_{\partial\Sigma} \delta A \delta c$,

$$\underline{S} = \frac{1}{2}\int_{\partial\Sigma} c \mathrm{d}_A c$$

- We can interpret this as an affine Poisson structure on $\Omega^1(\Sigma, \mathfrak{g})$, which we may regard as the dual of the affine Lie algebra $\hat{\mathfrak{g}} = \Omega^0(\Sigma, \mathfrak{g}) \oplus \mathbb{R}$. **P**
- This may be generalized to other field theories: e.g., 4d gravity.

Quantization of the example $n = 1$

- If $\partial\Sigma = \emptyset$, we expect to get a vector space $\mathcal{H}$ by geometric quantization of M together with a coboundary operator Ω quantizing S. Its cohomology in degree zero describes a quantization of the reduced phase space.**P**
- If $\partial\Sigma \neq \emptyset$, we expect $\mathcal{H}$ to be a representation of a quantization of $\underline{M}$.**P**
- For example, we may consider the deformation quantization ot the Poisson structure described by $\underline{M}$.**P**
- If $\Sigma = \Sigma_1 \cup_D \Sigma_2$ with D a common boundary component for Σ_1 and Σ_2, we expect $\mathcal{H}$ for Σ to be recovered as the tensor product of the $\mathcal{H}$s for Σ_1 and Σ_2 over the algebra associated to D.

Quantization of the example $n = 0$

Notation:
$\mathcal{F} = M$, the BV space of bulk field with data (ω, S, Q)
$\mathcal{F}^\partial = \underline{M}$, the BFV space of boundary fields with data $(\omega^\partial, S^\partial, Q^\partial)$ **P**

- In this case, we expect to get a vector space $\mathcal{H}$ by geometric quantization of $\mathcal{F}^\partial$ together with a coboundary operator Ω quantizing S^∂. We expect the gauge fixed integral of $e^{\frac{i}{\hbar}S}$ to yield an Ω-closed state (defined up to Ω-exact terms).**P**
- If $\Sigma = \Sigma_1 \cup_D \Sigma_2$ with D a common boundary component for Σ_1 and Σ_2, we expect the state for Σ to be recovered as the pairing of the states for Σ_1 and Σ_2 in the Hilbert space associated to D. **P**
- We produced a rather general construction, which relies on the existence of a “nice” polarization of $\underline{M}$. The construction also keeps track of “residual fields" (e.g., zero modes).**P**
- Main assumption $\omega^\partial = \mathrm{d}\omega^\partial$.

The polarization

- Assume we have an involutive Lagrangian distribution $\mathcal{P}$ on $\mathcal{F}^\partial$, called a polarization, such that the restriction of α^∂ to its leaves is zero. We may use gauge transformations to adapt α^∂.
- For simplicity we assume $\mathcal{B} := \mathcal{F}^\partial/\mathcal{P}$ to be smooth. **P**
- The crucial assumption now is that we have a splitting

$$\mathcal{F} = \mathcal{Y} \times \mathcal{B}$$

such that the BV form ω only has components along $\mathcal{Y}$ and is constant on $\mathcal{B}$. (A splitting is always possible locally; the crucial condition is on ω.)

Remark

In the infinite dimensional case (e.g., in field theory), it is possible to have a nondegenerate ω with this property. In the finite-dimensional case (e.g., in a discretized field theory), ω is then necessarily degenerate, but we still require it to be nondegenerate on $\mathcal{Y}$, which is enough to define BV integration.

The modified quantum master equation I

Using the splitting, we rewrite the mCME as (we no longer write π^*)

$$\delta_{\mathcal{Y}} S = \iota_{Q_{\mathcal{Y}}} \omega$$
$$\delta_{\mathcal{B}} S = -\alpha^{\partial} \mathbf{P}$$

The two equations imply

$$\frac{1}{2}(S, S)_{\mathcal{Y}} = \frac{1}{2} \iota_{Q_{\mathcal{Y}}} \iota_{Q_{\mathcal{Y}}} \omega = S^{\partial} \qquad (*)\mathbf{P}$$

Now assume we have adapted Darboux coordinates (b, p) on $\mathcal{F}^{\partial}$ with b on $\mathcal{B}$, p on the leaves and $\alpha^{\partial} = -\sum p \, \delta b$. Then the second equation implies

$$\frac{\delta S}{\delta b} = p \qquad (**)$$

This means that, in this splitting, S is linear in the b's.

The modified quantum master equation II

We now assume that S also solves the equation

$$\Delta_{\mathcal{Y}} S = 0$$

Remark

Without boundary this means that we assume that S solves both the classical and the quantum master equation. With boundary, Δ makes sense only on the $\mathcal{Y}$-factor. We will return on this. **P**

We then have

$$\Delta_{\mathcal{Y}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S} = \left(\frac{\mathrm{i}}{\hbar}\right)^2 \frac{1}{2} (S, S)_{\mathcal{Y}}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S}$$

and equation (*) implies

$$-\hbar^2 \Delta_{\mathcal{Y}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S} = S^{\partial} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S} \qquad (\dagger)$$

The modified quantum master equation III

We now move to the quantization. We take $\mathcal{H}$ to be an appropriate space of functions on $\mathcal{B}$.
Equation (**) essentially says that

$$\hat{p}S = -\mathrm{i}\hbar p \quad \text{with} \quad \hat{p} = -\mathrm{i}\hbar \frac{\delta}{\delta b}\mathbf{P}$$

Remark

Here S is an element of $\mathcal{H}$ parametrized by $\mathcal{Y}$. The p appearing in the equation is now an element of $\mathcal{Y}$. **P**

If we quantize S^∂ by the Schrödinger prescription

$$\Omega := S^\partial\left(b, -\mathrm{i}\hbar\frac{\delta}{\delta b}\right)$$

with all derivatives placed to the right, we get

$$\Omega \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} = S^\partial\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} \qquad (\ddagger)$$

The modified quantum master equation IV

Putting (†) and (‡) together we finally get the modified quantum master equation (mQME)

$$\boxed{(\hbar^2\Delta_{\mathcal{Y}} + \Omega)e^{\frac{i}{\hbar}S} = 0}$$

Remark

The assumption $\Delta_{\mathcal{Y}} S = 0$ is not really necessary (and is often not justified). More generally, we have

$$\Delta_{\mathcal{Y}} e^{\frac{i}{\hbar}S} = \left(\left(\frac{i}{\hbar}\right)\Delta_{\mathcal{Y}} S + \left(\frac{i}{\hbar}\right)^2 \frac{1}{2}(S,S)_{\mathcal{Y}}\right) e^{\frac{i}{\hbar}S}$$

If we define

$$S^{\partial}_{\hbar} := \frac{1}{2}(S,S)_{\mathcal{Y}} - i\hbar\Delta_{\mathcal{Y}} S = S^{\partial} + O(\hbar)$$

and Ω as the Schrödinger quantization of $S^{\partial}_{\hbar}$, we recover the mQME.

The modified quantum master equation V

By construction we have

$$\Delta_{\mathcal{Y}}^2 = 0 \qquad [\Delta_{\mathcal{Y}}, \Omega] = 0$$

The operator

$$\Omega_{\mathcal{Y}} := \hbar^2 \Delta_{\mathcal{Y}} + \Omega$$

appearing in the mQME then squares to zero iff

$$\Omega^2 = 0$$

The existence of a splitting such that this holds is a fundamental condition (absence of anomalies) which allows passing to the $\Omega_{\mathcal{Y}}$-cohomology. Cohomology in degree zero describes $\mathcal{Y}$-parametrized physical states.

The quantum state

- Assume the mQME
$$\Omega_{\mathcal{Y}} e^{\frac{i}{\hbar}S} = 0$$
- Suppose $\mathcal{Y} = \mathcal{Y}' \times \mathcal{Y}''$ (possibly $\mathcal{Y}'$ a point).
- Pick a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{Y}''$.
- Define
$$\psi := \int_{\mathcal{L}} e^{\frac{i}{\hbar}S} \in \mathcal{H} \otimes C^\infty(\mathcal{Y}')\mathbf{P}$$
- Then
  1. We have the induced mQME
  $$\Omega_{\mathcal{Y}'}\psi = 0$$
  2. Changing the "gauge fixing" $\mathcal{L}$ changes ψ by an $\Omega_{\mathcal{Y}'}$-exact term. **P**
- Hence ψ defines a $\Omega_{\mathcal{Y}'}$-cohomology class (of degree 0).
- We might iterate this procedure ("Wilson renormalization with boundary") and eventually arrive at $\mathcal{Y}'$ a point. In this case, ψ will be an Ω-cohomology class of degree zero on $\mathcal{H}$: a physical state.

Some results

- With Mnëv and Reshetikhin, we have applied the general formalism for the abelian *BF* theories, obtaining the state for every manifold with boundary and proving the gluing properties.**P**
- By perturbing abelian *BF* theory, we have extended the construction to other theories like
  1. Quantum mechanics and topological quantum mechanics
  2. Split Chern–Simons theory (also with Mnëv and Wernli)
  3. 2D Yang–Mills theory (developed in full details by Mnev and Iraso)
  4. Poisson sigma model (also with Moshayedi and Wernli)**P**
- In the last example, one can e.g. recover the associativity of Kontsevich's star product from the composition of states.**P**
- 2D Yang–Mills theory has been studied also for manifolds with corners. This way, one may recover the full nonperturbative results out of the perturbative expansions.

Final remarks

- Other theories like scalar field, spinor field, Yang–Mills can be treated alike (but one has to take renormalization into account). The case of 2d scalar fields has been recently studied in full detail by Kandel, Mnëv and Wernli. **P**
- Classical BV-BFV gravity theories (in the Einstein–Hilbert as well as in the Palatini–Cartan version) have also been studied.**P**
- A discretized version of nonabelian *BF* theory has also been studied: in this setting, all spaces are finite dimensional and all the quantum BV-BFV results are rigorous from the start. **P**

Thanks